$$\frac{\partial \vec{M}}{\partial t} = \gamma \vec{M} \times \vec{H} - \frac{(\vec{M} - \vec{M}_0)}{T}$$ Bloch Equations (1)

$H = (\cos \omega t\, H_1 \hat{x}, -\sin \omega t\, H_1 \hat{y}, H_0 \hat{z})$ Circular Polarization

$$M \times H|_x = M_y H_z - M_z H_y$$

$$M \times H|_y = M_z H_x - M_x H_z$$

$$M \times H|_z = M_x H_y - M_y H_x$$

$$\vec{M}_0 = M_0 \hat{z}$$

$$\frac{\partial \vec{M}}{\partial t} \cdot \hat{x} = \frac{\partial M_x}{\partial t}$$

$$\frac{\partial M_x}{\partial t} = \gamma (M_y H_z - M_z H_y) - \frac{M_x}{T_2}$$

$$\frac{\partial M_y}{\partial t} = \gamma (M_z H_x - M_x H_z) - \frac{M_y}{T_2}$$

$$\frac{\partial M_z}{\partial t} = \gamma (M_x H_y - M_y H_x) - \frac{M_z - M_0}{T_1}$$

$$\frac{\partial M_+}{\partial t} = -i\omega_0 M_+ + i\omega_1 M_z(\cos\omega t - i\sin\omega t) - \frac{M_+}{T_2}$$

$$\frac{\partial M_-}{\partial t} = i\omega_0 M_- - i\omega_1 M_z(\cos\omega t + i\sin\omega t) - \frac{M_-}{T_2}$$

$$\frac{\partial M_z}{\partial t} = -M_x\omega_1\sin\omega t - M_y\omega_1\cos\omega t - \frac{M_z - M_0}{T_1}$$

$$= -\frac{M_+ + M_-}{2}\omega_1\sin\omega t + i\frac{(M_+ - M_-)}{2}\cos\omega t - \frac{M_z - M_0}{T_1}$$

$$= i\frac{M_+\omega_1}{2}(\cos\omega t + i\sin\omega t) - iM_-\frac{\omega_1}{2}(\cos\omega t - i\sin\omega t)$$

//

$$\frac{\partial M_+}{\partial t} = -i\omega_0 M_+ + i\omega_1 M_z e^{-i\omega t} - \frac{M_+}{T_2}$$

$$\frac{\partial M_-}{\partial t} = i\omega_0 M_- - i\omega_1 M_z e^{+i\omega t} - \frac{M_-}{T_2}$$

$$\frac{\partial M_z}{\partial t} = iM_+\frac{\omega_1}{2}e^{+i\omega t} - iM_-\frac{\omega_1}{2}e^{-i\omega t} - \frac{M_z - M_0}{T_1}$$

$$M_\pm \equiv \tilde{M}_\pm e^{\mp i\omega t} \Rightarrow \frac{\partial M_\pm}{\partial t} = \frac{\partial\tilde{M}_\pm}{\partial t}e^{\mp i\omega t} \mp i\omega\tilde{M}_\pm e^{\mp i\omega t}$$

① $$\frac{\partial \tilde{M}_+}{\partial t} e^{-i\omega t} = i(\omega-\omega_0)\tilde{M}_+ e^{-i\omega t} + i\omega_1 \tilde{M}_z e^{-i\omega t} - \tilde{M}_+ \frac{e^{-i\omega t}}{T_2}$$

② $$\frac{\partial \tilde{M}_-}{\partial t} e^{i\omega t} = -i(\omega-\omega_0)\tilde{M}_- e^{i\omega t} - i\omega_1 \tilde{M}_z e^{i\omega t} - \tilde{M}_- \frac{e^{i\omega t}}{T_2}$$

③ $$\frac{\partial \tilde{M}_z}{\partial t} = i\frac{M_+}{2}\omega_1 e^{i\omega t} - i\frac{M_-}{2}\omega_1 e^{-i\omega t} - \frac{\tilde{M}_z - M_0}{T_1}$$

$$= i\tilde{M}_+\frac{\omega_1}{2} - i\tilde{M}_-\frac{\omega_1}{2} - \frac{\tilde{M}_z - M_0}{T_1}$$

Muliply ① by $\frac{1}{\sqrt{2}} e^{i\omega t}$ and ② by $\frac{1}{\sqrt{2}} e^{-i\omega t}$

and let $m_\pm = \frac{\tilde{M}_\pm}{\sqrt{2}}$ $m_z = \tilde{M}_z$ $m_0 = M_0$ $\Delta\omega = \omega - \omega_0$

Then:

$$\frac{\partial m_+}{\partial t} = \left[i\Delta\omega - \frac{1}{T_2}\right] m_+ + 0\cdot m_- + i\frac{\omega_1}{\sqrt{2}} m_z + 0$$

$$\frac{\partial m_-}{\partial t} = 0\cdot m_+ + \left[-i\Delta\omega - \frac{1}{T_2}\right] m_- - i\frac{\omega_1}{\sqrt{2}} m_z + 0$$

$$\frac{\partial m_z}{\partial t} = \frac{i\omega_1}{\sqrt{2}} m_+ - \frac{i\omega_1}{\sqrt{2}} m_- - \frac{m_z}{T_1} + \frac{m_0}{T_1}$$

In matrix notation, we have:

$$\frac{\partial}{\partial t}\begin{bmatrix} m_+ \\ m_- \\ m_z \end{bmatrix} = \begin{bmatrix} (i\Delta\omega - \frac{1}{T_2}) & 0 & \frac{i\omega_1}{\sqrt{2}} \\ 0 & (-i\Delta\omega - \frac{1}{T_2}) & -\frac{i\omega_1}{\sqrt{2}} \\ \frac{i\omega_1}{\sqrt{2}} & -\frac{i\omega_1}{\sqrt{2}} & -\frac{1}{T_1} \end{bmatrix}\begin{bmatrix} m_+ \\ m_- \\ m_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m_0/T_1 \end{bmatrix}$$

In the steady state, after all the transients have died out $\frac{\partial m_\pm}{\partial t}, \frac{\partial m_z}{\partial t} = 0$

$$\begin{bmatrix} -i\Delta\omega + \frac{1}{T_2} & 0 & \frac{-i\omega_1}{\sqrt{2}} \\ 0 & i\Delta\omega + \frac{1}{T_2} & \frac{i\omega_1}{\sqrt{2}} \\ \frac{-i\omega_1}{\sqrt{2}} & \frac{i\omega_1}{\sqrt{2}} & \frac{1}{T_1} \end{bmatrix}\begin{bmatrix} m_+ \\ m_- \\ m_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m_0/T_1 \end{bmatrix}$$

which is in the form

(4) $$\underset{\sim}{W}(\Delta\omega)\cdot\underset{\sim}{m} = m_0/T_1\cdot\underset{\sim}{\eta} \qquad \eta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \underset{\sim}{m} = \begin{bmatrix} m_+ \\ m_- \\ m_z \end{bmatrix}$$

$$\underset{\sim}{W}(\Delta\omega) \equiv \begin{bmatrix} a & 0 & -id \\ 0 & a^* & id \\ -id & id & b \end{bmatrix}$$

$a = -i\Delta\omega + \frac{1}{T_2}$

$a^* = i\Delta\omega + \frac{1}{T_2}$

$b = \frac{1}{T_1}$

$d = \frac{\omega_1}{\sqrt{2}}$

Solve (4) by matrix inversion

$$\underset{\sim}{m} = \underset{\sim}{W}(\Delta\omega)^{-1} \frac{m_0}{T_1} \underset{\sim}{\eta}$$

To find the inverse of $\underset{\sim}{W}(\Delta\omega)$ we need the

determinant $|\underset{\sim}{W}(\Delta\omega)| \equiv W = a a^* b + a d^2 + a^* d^2$

$$\therefore \underset{\sim}{W}(\Delta\omega)^{-1} = \frac{1}{W} \begin{bmatrix} a^* b + d^2 & d^2 & i a^* d \\ d^2 & ab + d^2 & -iad \\ i a^* d & -iad & aa^* \end{bmatrix}$$

evaluated by trial and error

$$\underset{\sim}{m} = \frac{m_0/T_1}{a a^* b + (a + a^*) d^2} \begin{bmatrix} i a^* d \\ -iad \\ aa^* \end{bmatrix}$$

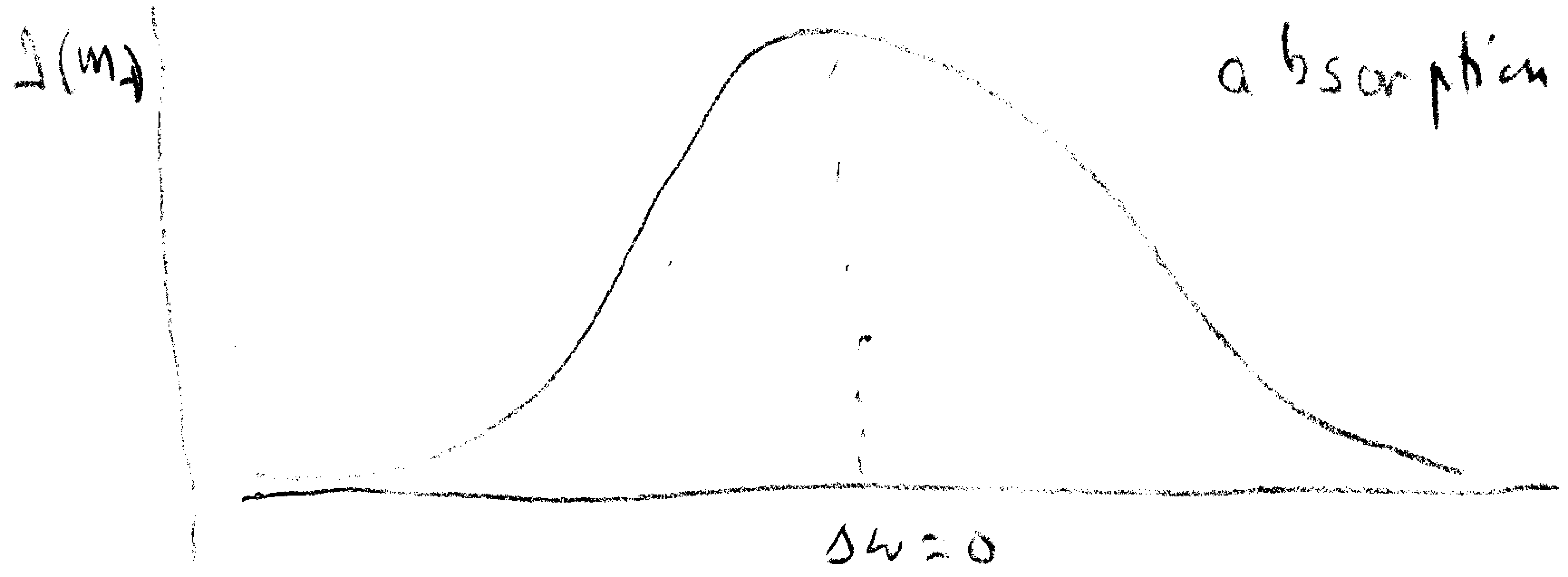
$$m_+ = \frac{m_0/T_1 \left(-\Delta\omega + \frac{i}{T_2}\right) \frac{\omega_1}{\sqrt{2}}}{\left[(\Delta\omega)^2 + \left(\frac{1}{T_2}\right)^2\right] \frac{1}{T_1} + \frac{2}{T_2} d^2}$$

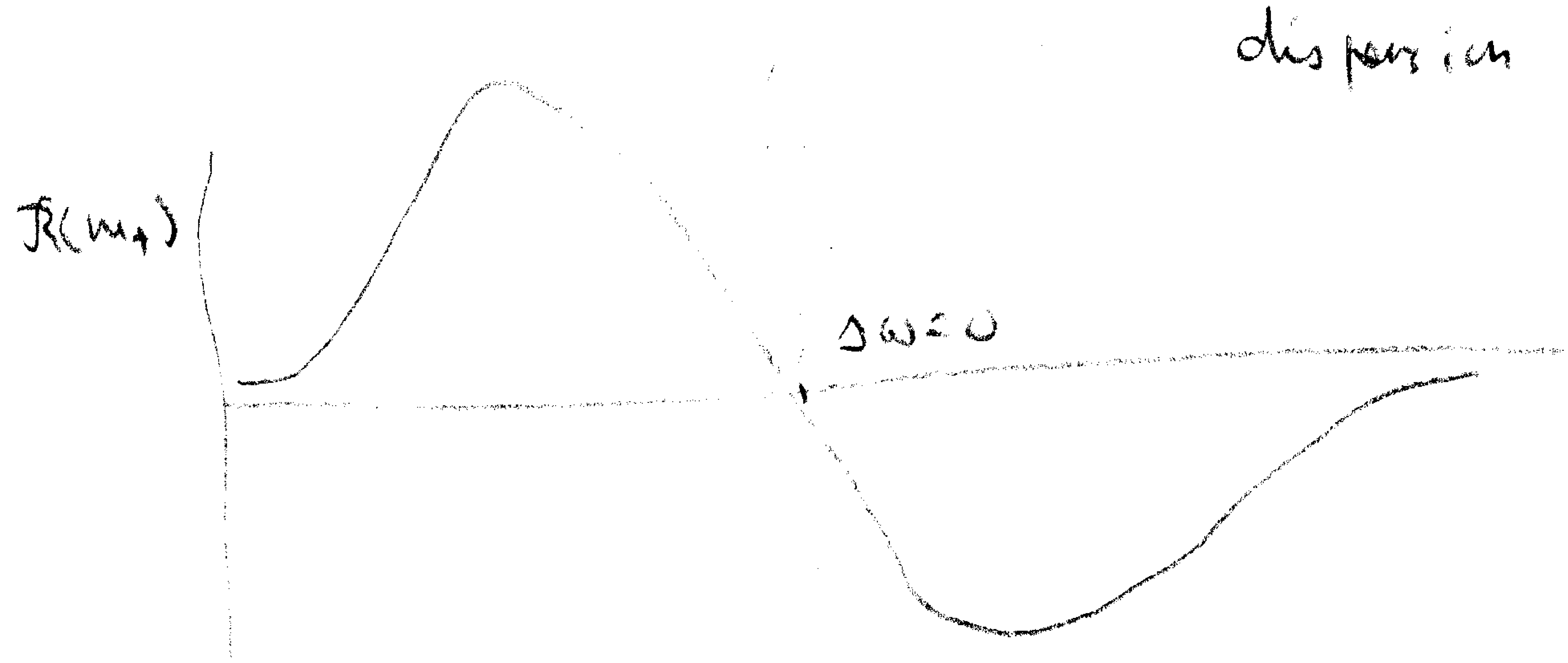
dispersion

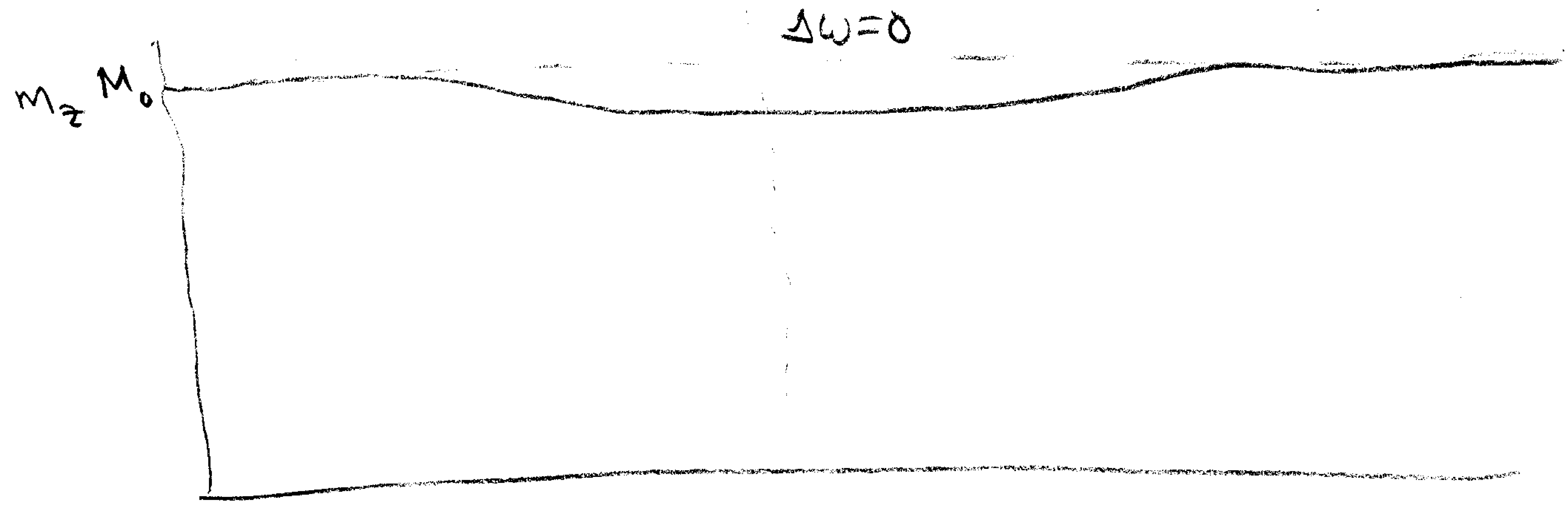
absorption

$$M_+ = \frac{M_0 (\omega_1 T_2)(-\Delta\omega T_2 + i)}{1 + (\Delta\omega T_2)^2 + T_1 T_2 \omega_1^2}$$

$$M_z = \frac{(M_0/T_1) \cdot (\Delta\omega^2 + (1/T_2)^2)}{\left[(\Delta\omega)^2 + \left(\frac{1}{T_2}\right)^2\right] 1/T_1 + \frac{2d^2}{T_2}} = \frac{M_0 \left[(\Delta\omega T_2)^2 + 1\right]}{1 + (\Delta\omega T_2)^2 + T_1 T_2 \omega_1^2}$$


ℑ(m₊)
absorption
Δω=0


dispersion
ℜ(m₊)
Δω=0


Δω=0
M_z M_0

$H = [2H_1 \cos\omega t,\ 0,\ H_0]$ Linear polarization case (1)

$$\frac{\partial \vec{M}}{\partial t} = \gamma \vec{M} \times \vec{H} - \frac{\vec{M} - \vec{M}_0}{T}$$ Bloch equations

$$\frac{\partial M_x}{\partial t} = \gamma \left[M_y H_z - M_z \cancel{H_y}^{\,0} \right] - \frac{M_x}{T_2}$$

$$\frac{\partial M_y}{\partial t} = \gamma \left[M_z H_x - M_x H_z \right] - \frac{M_y}{T_2}$$

$$\frac{\partial M_z}{\partial t} = \gamma \left[M_x \cancel{H_y}^{\,0} - M_y H_x \right] - \frac{M_z - M_0}{T_1}$$

$$\frac{\partial M_x}{\partial t} = -\frac{1}{T_2} M_x + \omega_0 M_y \qquad (\gamma H_z \equiv \omega_0)$$

$$\frac{\partial M_y}{\partial t} = -\omega_0 M_x - \frac{1}{T_2} M_y + 2\omega_1 \cos\omega t\, M_z \qquad (\gamma H_1 \equiv \omega_1)$$

$$\frac{\partial M_z}{\partial t} = -2\omega_1 \cos\omega t\, M_y - \frac{1}{T_1} M_z + \frac{M_0}{T_1}$$

Define $M_+ = M_x + i M_y$, $M_- = M_x - i M_y$

$$M_y = \frac{i}{2}(M_- - M_+) = \frac{i}{2}\left(M_x - iM_y - (M_x + iM_y)\right) = \frac{i}{2}(-2iM_y)$$

$$\overset{\checkmark}{=} M_y$$

$$\frac{\partial M_+}{\partial t} = \frac{\partial M_x}{\partial t} + i\frac{\partial M_y}{\partial t}$$

$$= -\frac{1}{T_2} M_x - i\omega_0 \cdot i M_y$$

$$- i\omega_0 M_x - \frac{1}{T_2} \cdot i M_y + 2i\omega_1 \cos\omega t \, M_z$$

$$= -\left(\frac{1}{T_2} + i\omega_0\right) M_+ + 0 \cdot M_- + 2i\omega_1 \cos\omega t \, M_z$$

$$\frac{\partial M_-}{\partial t} = \frac{\partial M_x}{\partial t} - i\frac{\partial M_y}{\partial t}$$

$$= -\frac{1}{T_2} M_x + i\omega_0(-i M_y)$$

$$+ i\omega_0 M_x - \frac{1}{T_2}(-i M_y) - 2i\omega_1 \cos\omega t \, M_z$$

$$= 0 \cdot M_+ - \left(\frac{1}{T_2} - i\omega_0\right) M_- - 2i\omega_1 \cos\omega t \, M_z$$

$$\frac{\partial M_z}{\partial t} = + i\omega_1 \cos\omega t \, M_+ - i\omega_1 \cos\omega t \, M_- - \frac{1}{T_1} M_z + \frac{M_0}{T_1}$$

$$\frac{\partial M_+}{\partial t} = -\left(\frac{1}{T_2} + i\omega_0\right) M_+ + 0 \cdot M_- + 2i\omega_1 \cos\omega t \, M_z$$

$$\frac{\partial M_-}{\partial t} = 0 \cdot M_+ - \left(\frac{1}{T_2} - i\omega_0\right) M_- - 2i\omega_1 \cos\omega t \, M_z$$

$$\frac{\partial M_z}{\partial t} = i\omega_1 \cos\omega t \, M_+ - i\omega_1 \cos\omega t \, M_- - \frac{1}{T_1} M_z + \frac{M_0}{T_1}$$

$$M_+ = \sum_{k=-\infty}^{\infty} m_+^{(k)} e^{ik\omega t}, \quad M_- = \sum_{k=-\infty}^{\infty} m_-^{(k)} e^{ik\omega t}, \quad M_z = \sum_{k=-\infty}^{\infty} m_z^{(k)} e^{ik\omega t}$$

$$M_0 = \sum_{k=-\infty}^{\infty} m_0^{(k)} e^{ik\omega t}$$

$$\frac{\partial M_+}{\partial t} = \sum_{k=-\infty}^{\infty} \left(\frac{\partial m_+^{(k)}}{\partial t} + ik\omega\right) e^{ik\omega t}$$

These are known as Floquet expansions (it is assumed that the expansion converges)

$$\frac{\partial M_-}{\partial t} = \sum_{k=-\infty}^{\infty} \left(\frac{\partial m_-^{(k)}}{\partial t} + ik\omega\right) e^{ik\omega t}$$

$$\frac{\partial M_z}{\partial t} = \sum_{k=-\infty}^{\infty} \left(\frac{\partial m_z^{(k)}}{\partial t} + ik\omega\right) e^{ik\omega t}$$

$$\pm 2i\omega_1 \cos\omega t \, M_z = \pm i\omega_1 \sum_{k=-\infty}^{\infty} \left(e^{i\omega t} + e^{-i\omega t}\right) m_z^{(k)} e^{il\omega t}$$

$$= \pm i\omega_1 \sum_{k=-\infty}^{\infty} \left(m_z^{(k-1)} + m_z^{(k+1)}\right) e^{ik\omega t}$$

$$\pm i\omega_1 \cos\omega t \, M_\pm = \pm \frac{i\omega_1}{2} \sum_{k=-\infty}^{\infty} \left(e^{i\omega t} + e^{-i\omega t}\right) m_\pm^{(k)} e^{ik\omega t}$$

$$= \pm \frac{i\omega_1}{2} \sum_{k=-\infty}^{\infty} \left(m_\pm^{(k-1)} + m_\pm^{(k+1)}\right) e^{ik\omega t}$$

$$\sum_{k=-\infty}^{\infty} e^{ik\omega t}\left[\frac{\partial m_+^{(k)}}{\partial t} = -\left(\frac{1}{T_2} + ik\omega + i\omega_0\right)m_+^{(k)} + i\omega_1\left(m_z^{(k-1)} + m_z^{(k+1)}\right)\right]$$

$$\sum_{k=-\infty}^{\infty} e^{ik\omega t}\left[\frac{\partial m_-^{(k)}}{\partial t} = -\left(\frac{1}{T_2} + ik\omega - i\omega_0\right)m_-^{(k)} - i\omega_1\left(m_z^{(k-1)} + m_z^{(k+1)}\right)\right]$$

$$\sum_{k=-\infty}^{\infty} e^{ik\omega t}\left[\frac{\partial m_z^{(k)}}{\partial t} = \frac{i\omega_1}{2}\left(m_+^{(k-1)} + m_+^{(k+1)}\right) - \frac{i\omega_1}{2}\left(m_-^{(k-1)} + m_+^{(k+1)}\right) - \frac{1}{T_1}m_z^{(k)} + \frac{1}{T_1}m_0^{(k)}\right]$$

$$\int_{-2\pi/\omega}^{2\pi/\omega} dt\, e^{-in\omega t} e^{ik\omega t} = \int_{-2\pi/\omega}^{2\pi/\omega} dt\, e^{i(k-n)\omega t}$$

A useful integral

$$= \begin{cases} \dfrac{1}{i(k-n)}\, e^{i(k-n)\omega t}\Big|_{-2\pi/\omega}^{2\pi/\omega}, & k \neq n \\[2ex] 4\pi/\omega, & k = n \end{cases}$$

$$e^{i(k-n)\omega t}\Big|_{-2\pi/\omega}^{2\pi/\omega} = e^{i(k-n)2\pi} - e^{-i(k-n)2\pi}$$

But $k-n$ is an integer by construction, so

$$e^{\pm i(k-n)2\pi} = 1$$

$$\therefore \int_{-2\pi/\omega}^{2\pi/\omega} dt\, e^{i(k-n)\omega t} = \begin{cases} 4\pi/\omega, & k = n \\ 0, & k \neq n \end{cases}$$

Operate from the left with $\dfrac{\omega}{4\pi}\displaystyle\int_{-2\pi/\omega}^{2\pi/\omega} dt\, e^{-in\omega t}$

selects out the n^{th} Floquet mode.

$$\frac{\partial m_+^{(n)}}{\partial t} = -\left(\frac{1}{T_2} + i(n\omega + \omega_0)\right) m_+^{(n)} + i\omega_1 \left(m_z^{(n-1)} + m_z^{(n+1)}\right)$$

$$\frac{\partial m_-^{(n)}}{\partial t} = -\left(\frac{1}{T_2} + i(n\omega - \omega_0)\right) m_-^{(n)} - i\omega_1 \left(m_z^{(n-1)} + m_z^{(n+1)}\right)$$

$$\frac{\partial m_z^{(n)}}{\partial t} = \frac{i\omega_1}{2}\left[m_+^{(n-1)} - m_-^{(n-1)}\right] + \frac{i\omega_1}{2}\left[m_+^{(n+1)} - m_-^{(n+1)}\right] - \frac{1}{T_1}\left(m_z^{(n)} - m_0^{(n)}\right)$$

This is a coupled set of equations of infinite order.
Need to develop a criterion for truncating the set.

Consider the equations for $n \in \{-1, 0, 1\}$

$$\frac{\partial m_+^{(-1)}}{\partial t} = -\left(\frac{1}{T_2} - i(\omega - \omega_0)\right) m_+^{(-1)} + i\omega_1 \left(m_z^{(-2)} + m_z^{(0)}\right)$$

$$\frac{\partial m_-^{(1)}}{\partial t} = -\left(\frac{1}{T_2} + i(\omega - \omega_0)\right) m_-^{(1)} - i\omega_1 \left(m_z^{(0)} + m_z^{(2)}\right)$$

$$\frac{\partial m_z^{(0)}}{\partial t} = \frac{i\omega_1}{2}\left[m_+^{(-1)} - m_-^{(-1)}\right] + \frac{i\omega_1}{2}\left[m_+^{(1)} - m_-^{(1)}\right] - \frac{1}{T_1}\left(m_z^{(0)} - m_0^{(0)}\right)$$

Note, the equations of motion for $m_+^{(1)}, m_-^{(-1)}$ have large coefficients proportional (approximately) to $\pm 2\omega_0$.
As ω_0 is typically much larger than $\frac{1}{T_2} \sim \frac{1}{T_1} \sim \omega_1$
$m_\pm^{(\pm 1)}$ must be smaller than $m_z^{(0)}$ by a factor of $\omega_1/2\omega_0$
in order to have a consistent set of equations.
By a similar line of reasoning $m_z^{(2)}$ will also be smaller than $m_z^{(0)}$ by $\sim \omega_1/\omega_0$.
At low fields when $\omega_1 \sim \omega_0$ more Floquet terms need to be kept.

On this basis, we can develop a consistent high field, moderate saturation approximation by retaining only the $m_+^{(-1)}$, $m_-^{(1)}$, $m_z^{(0)}$, $m_0^{(0)}$ terms

One finds:

$$\frac{\partial m_+^{(-1)}}{\partial t} = -\left(\frac{1}{T_2} - i(\omega-\omega_0)\right) m_+^{(-1)} + i\omega_1 m_z^{(0)}$$

$$\frac{\partial m_-^{(1)}}{\partial t} = -\left(\frac{1}{T_2} + i(\omega-\omega_0)\right) m_-^{(1)} - i\omega_1 m_z^{(0)}$$

$$\frac{\partial m_z^{(0)}}{\partial t} = \frac{i\omega_1}{2} m_+^{(-1)} - \frac{i\omega_1}{2} m_-^{(1)} - \frac{1}{T_1}\left(m_z^{(0)} - m_0^{(0)}\right)$$

Define $m_+ = \frac{m_+^{(-1)}}{\sqrt{2}}$, $m_- = \frac{m_-^{(1)}}{\sqrt{2}}$, $m_z = m_z^{(0)}$, $m_0^{(0)} = 0$

$$\frac{\partial m_+}{\partial t} = -\left(\frac{1}{T_2} - i(\omega-\omega_0)\right) m_+ + 0 + \frac{i\omega_1}{\sqrt{2}} m_z + 0$$

$$\frac{\partial m_-}{\partial t} = 0 - \left(\frac{1}{T_2} + i(\omega-\omega_0)\right) m_- - \frac{i\omega_1}{\sqrt{2}} m_z + 0$$

$$\frac{\partial m_z}{\partial t} = \frac{i\omega_1}{\sqrt{2}} m_+ - \frac{i\omega_1}{\sqrt{2}} m_- - \frac{1}{T_1} m_z + \frac{1}{T_1} m_0$$

But we've solved this case already.

In the high field, moderate saturation approximation it doesn't matter whether one uses linearly or circularly polarized excitation.